NUMERICAL PROPERTIES OF A BOUNDARY CROSSING PROCESS USEFUL IN LIFE TESTING

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THESIS

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CROSSING PROCESS USEFUL IN LIFE TESTING

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ABSTRACT

An approximate boundary for a finite sample sequential decision process is presented (without derivation). By means of Computer Simulation, properties of the process are checked; the power of the test is thus determined for selected alternative hypotheses.

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I. INTRODUCTION

The main purpose of this paper is to study the properties of a mathematically derived sequential sampling procedure. The derivation involves the assumption that a certain parameter N, identified as a sample size, was "large." Our purpose was to make computer calculations that would reveal how the sampling procedure might behave for finite, or even rather small, N. Other calculations would also indicate the power characteristics of the test, that is, the probability of rejecting a null hypothesis. The null hypothesis, in this case, was that a particular probability distribution describes the sample. By experimental sampling methods we attempted to estimate the power that the procedure possessed to detect departures of various sorts from the null hypothesis.

The stochastic process used to obtain the sequential procedure might also be used for other purposes in operations research. For example, the sequential procedure might be used to describe arrivals in <u>transitory queueing</u> situations. Some of these situations are briefly described at the end of this paper.

II. A SEQUENTIAL SAMPLING PROBLEM

Suppose, for example, that a supplier of transistors asserted that each of his transistors had an in-service lifetime T, where T was a random variable distributed exponentially;

$$F(t) = \begin{cases} 1 - e^{-\lambda_o t}, & t \geq 0 \\ 0, & t < 0 . \end{cases}$$

In order to test that assertion or hypothesis, a sample of N transistors was selected and placed in service simultaneously at time $t = 0$. On the average, the number of transistors that failed by time $t > 0$ was $NF(t)$, i.e.- if $A_N(t)$ denoted the number of transistors that failed by time t, then

$$E\left[A_N(t)\right] = NF(t)$$

and

$$\mathrm{Var}\left[A_N(t)\right] = NF(t)\left[1-F(t)\right],$$

since the transistors were presumed to fail, or die, independently.

If one waited until all N transistors had failed, the rate could be estimated and compared to λ_o. A hypothesis test can be conducted by waiting until r out of N transistors failed ($r < N$); this was shown by Epstein and Sobel [1].

The present approach to this problem was to establish a boundary, $B_N(t)$, with the property that if $\lambda = \lambda_o$, i.e. if the supplier was entirely honest, then $A_N(t) < B_N(t)$ for <u>all</u> t, with a known, specifiable, probability denoted by $1-\delta$. For instance, suppose $\delta = 0.05$. Then, if $\lambda = \lambda_o$, the probability that the accumulated numbers of failures by

time t would ever exceed the boundary is five per cent. Hence, if A_N remains below the boundary, $\lambda \leq \lambda_0$, with ninety-five per cent confidence.

The detailed derivation of the boundary is given by Gaver, Lehoczky, and Perlas in an unpublished technical report that is in preparation. The result is quoted below.

The Boundary

If F(t) is sufficiently smooth (possesses a continuous density) and if

$$B_N(t) = NF(t) + \sqrt{N}\left\{\alpha F(t) + \beta\left[1-F(t)\right]\right\}, \tag{2.1}$$

then the probability that $A_N(t) < B_N(t)$ for all t is approximately $1-e^{-2\alpha\beta}$; the approximation improving as N becomes large ($N \to \infty$). Here, α and β are arbitrary positive numbers.

A graph of a possible failure arrival pattern is a step function with unit increases at time points that are the order statistics of F(t): $T_{(1)}, T_{(2)}, \ldots$ where $T_{(1)}$ is the smallest of the N failure times, $T_{(2)}$ is the next smallest, ... $T_{(k)}$ is the kth smallest. Then, a particular sample may be represented as follows:

$$\begin{aligned}
A_N(t) &= 0 \quad \text{for} \quad 0 \leq t < T_{(1)} \\
&= 1 \quad \text{for} \quad T_{(1)} \leq t < T_{(2)} \\
&= 2 \quad \text{for} \quad T_{(2)} \leq t < T_{(3)} \\
&\quad \cdots \\
&= k \quad \text{for} \quad T_{(k)} \leq t < T_{(k+1)} \\
&\quad \cdots \\
&= N \quad \text{for} \quad T_{(N)} \leq t ,
\end{aligned} \tag{2.2}$$

and graphically, we have the following picture.

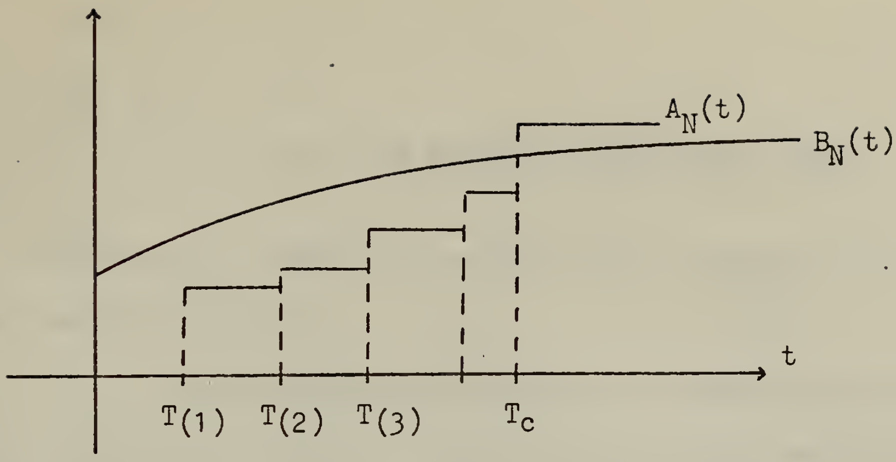

Figure 1.

In Figure 1, a crossing of the boundary occurs at T_c. We would conclude that the null hypothesis is not satisfied if such an event occurs. Notice that the parameters α and β may be selected at will. It is interesting to arrange that $\alpha\beta = k$, a constant such that $e^{-2\alpha\beta} = e^{-2k} = 0.05$, for example, and then to attempt to select α (or β) so that the boundary will be crossed soon if a non-null distribution of specific sort is to be detected [2]. This problem has not been extensively investigated, but the expected time to hit the boundary has been tabulated for certain of the problems studied.

III. USE OF THE COMPUTER SIMULATION

Several facts about the boundary $B_N(t)$ presented in equation (2.1) are worthy of notice.

a. The prescribed probability of crossing the boundary, $e^{-2\alpha\beta}$, is not exact for finite N. That is, use of this probability to assess the probability of crossing is an approximation which is, in fact, quite analogous to the use of the normal distribution as an approximation to the distribution of a sum of independent random variables. The mathematical derivation does not indicate whether the actual probability of crossing is, perhaps, consistently lower or higher than the actual true probability. More complex mathematical methods may provide useful information of this type, but a straightforward approach is that of simulation or synthetic sampling. By this means, it can be seen how increases in N improve the quality of our approximation, as is suggested by the mathematical derivation.

b. The relationship of the true crossing probability to the approximate crossing probability, $e^{-2\alpha\beta}$ for given N, may possibly be influenced by the choice of α and β. Setting $\alpha\beta = k$, where k is a constant delivering a desired probability of crossing (for 5%, $k \approx 1.49$), α (and hence β) may be adjusted so as to obtain good agreement between the theory ($e^{-2k} = e^{-2\alpha\beta}$) and the true probability for various values of N-- in particular for small N. In this paper, an attempt was made to study this problem by simulation.

In order to study the properties of the boundary $B_N(t)$, the following simulation was carried out.

A. SIMULATION LOGIC

1. The null hypothesis distribution was chosen and called $F_0(t)$. The formula used to calculate the boundary values was as follows:

$$B_N(t) = NF_0(t) + \sqrt{N}\ \{\alpha F_0(t) + \beta\ [1-F_0(t)]\}\ . \tag{3.1}$$

An exponential distribution is the null hypothesis, as in the illustrative problem; hence, for the initial calculations, the equation

$$B_N(t) = N(1-e^{-t}) + \sqrt{N}\ \{\beta + (\alpha - \beta)\ (1-e^{-t})\} \tag{3.2}$$

was used.

2. A sampling distribution was selected. Initially, it was the null hypothesis distribution, so that the accuracy of the theoretical estimate of the probability of crossing the boundary could be checked. Later, other distributions were tried in order to determine the power for departures from the exponential distribution.

3. A sample of size N was taken and arranged in order:

$$t_{(1)}<t_{(2)}<t_{(3)}\cdots<t_{(k)}<\cdots t_{(N)}. \tag{3.3}$$

This ordered sample was composed of the actual numbers obtained as realizations of the random variables.

$$T_{(1)}<T_{(2)}<\cdots T_{(k)}<\cdots T_{(N)}. \tag{3.4}$$

The details on how the random samples were conducted will be presented shortly.

4. At each failure time, say $t_{(k)}$, starting with k = 1, then 2, 3, ..., $A_N(t_{(k)}) = k$ was compared with $B_N(t_{(k)})$: if $k < B_N(t_{(k)})$, for k = 1, 2, ..., the process continued to k + 1 and was checked again. If at k^*, the event $k^* \geq B_N(t_{(k^*)})$ occurred _for the first time_, a boundary

crossing was recorded at the time $t_c \equiv t_{k*}$; the test was stopped and another simulation begun.

5. Step four, above, was independently repeated NR times (NR means the number of replications). The probability of crossing f_N was estimated by the fraction

$$f_N = \frac{\text{number of crossings.}}{NR} \tag{3.5}$$

Clearly, NRf_N was a realization of a binomially distributed random variable. An approximate idea of the variability of the estimate of the probability of crossing is given by

$$\hat{\sigma}^2_{NR} = \frac{f_N(1-f_N)}{NR} . \tag{3.6}$$

An approximate confidence limit for the true probability P_N of crossing came from assuming that f_N is approximately normal with variance $\hat{\sigma}^2_{NR}$.

B. SIMULATION PROGRAM

The simulation procedure just described was written in FORTRAN computer programming language and run on the IBM 360, Model 67 Computer at the Naval Postgraduate School. The program was written to facilitate changes of distributions, boundary parameters (α and β), and numbers of replications (NR). It was also constructed so that the sampling process could be stopped after a fixed time period, i.e.- before all of the order statistics had been recorded.

The pseudo random number generator employed to obtain realizations of uniform random numbers was a library subroutine called RANDOM [3 and 4]; it is based on a Lehmer multiplicative scheme. The uniform realizations were then transformed into other random numbers by a probability integral transformation.

A program listing and sample output has been included at the end of this paper. In order to generate the sequence of order statistics (interpreted as failure or arrival times), the random sample of size N was first created. Then, a SORT routine was employed to put the times in order. A more efficient procedure would have been to generate the ordered increments sequentially; for example, the time between $T_{(k)}$ and $T_{(k+1)}$ in exponential sampling is also exponential with a mean proportional to $\frac{1}{N-k}$. One program has been written in this vein, but production runs for the present investigation have largely been carried out by use of the SORT routine [5].

IV. SIMULATION EXPERIMENTS

Several simulation experiments have been carried out to develop an understanding of the boundary crossing process.

A. EXPERIMENT 1

This experiment explores the crossing probability under the null hypothesis ($\lambda = \lambda_0$) and under selected alternatives, for various sample sizes (N); and for different boundary parameters α and β , but with $\alpha = \beta$. The computed probability of a crossing f_N was calculated by the fraction (3.5) mentioned previously. The average number of arrivals up to a crossing ($\bar{A}(t)$) was calculated by summing the number of arrivals ($A_N(t)$) that had occurred up to the time of crossing, over the two thousand replications and dividing by the number of crossings (NC) recorded. The average crossing time ($\bar{C}(t)$) was calculated by adding the arrival times, which generated crossings, then dividing that sum by the number of crossings. The results appear in Table I (theoretical crossing probability of five per cent), and Table II (theoretical crossing probability of ten per cent). These Tables are based on two thousand replications (NR = 2000).

When the boundary was not crossed for a sample size N, all values for that N appear as 0. In such a situation, the zero (0) is intended as a meaningless place holder rather than the unique number.

TABLE I.

5% Probability of a Crossing
$\alpha = \beta = 1.22388$

EXPON (λ = .5)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	3	3.33	.133	0.15
6	2	5.00	.300	0.10
8	7	6.42	.428	0.35
10	1	8.00	.523	0.05
25	0	0.00	.000	0.00
50	0	0.00	.000	0.00
75	0	0.00	.000	0.00

EXPON (λ = 1)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	46	3.63	.287	2.30
6	69	5.07	.391	3.45
8	66	6.30	.402	3.30
10	70	7.37	.416	3.50
25	73	15.19	.461	3.65
50	85	27.54	.504	4.25
75	104	42.43	.586	5.20

EXPON (λ = 1.25)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	100	3.66	.298	5.00
6	141	5.11	.381	7.05
8	165	6.33	.414	8.25
10	197	7.48	.428	9.84
25	372	15.95	.513	18.59
50	644	28.64	.542	32.19
75	839	41.09	.560	41.95

TABLE I. (continued)

EXPON ($\lambda = 1.5$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	174	3.68	.303	8.70
6	253	5.03	.362	12.64
8	331	6.26	.395	16.54
10	404	7.48	.426	20.19
25	896	15.67	.498	44.79
50	1461	25.54	.470	73.04
75	1770	35.59	.432	88.49

EXPON ($\lambda = 1.75$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	262	3.70	.301	13.10
6	389	4.97	.349	19.45
8	525	6.19	.381	26.24
10	672	7.43	.418	33.59
25	1391	14.80	.437	69.54
50	1893	23.31	.362	94.64
75	1985	28.30	.274	99.24

EXPON ($\lambda = 2$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	360	3.69	.296	18.45
6	552	4.98	.346	27.59
8	741	6.15	.372	37.04
10	929	7.31	.397	46.44
25	1736	13.95	.382	86.79
50	1985	19.86	.256	99.24
75	2000	23.65	.190	100.00

TABLE II.

10% Probability of a Crossing
$\alpha = \beta = 1.07298$

EXPON (λ = .5)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	15	3.33	.300	0.75
6	9	5.00	.467	0.45
8	11	5.72	.341	0.55
10	5	6.80	.400	0.25
25	0	0.00	.000	0.00
50	0	0.00	.000	0.00
75	0	0.00	.000	0.00

EXPON (λ = 1)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	127	3.49	.388	6.35
6	151	4.90	.428	7.54
8	154	5.92	.427	7.70
10	156	7.03	.443	7.79
25	160	14.86	.495	8.00
50	177	26.74	.522	8.85
75	190	39.06	.551	9.50

EXPON (λ = 1.25)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	241	3.51	.345	12.05
6	289	4.72	.381	14.45
8	334	5.98	.425	16.70
10	374	7.10	.446	18.70
25	627	15.14	.517	31.34
50	945	26.86	.527	47.24
75	1168	38.66	.543	58.39

TABLE II. (continued)

EXPON ($\lambda = 1.5$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	360	3.54	.345	17.99
6	460	4.72	.376	22.79
8	571	5.85	.405	28.54
10	674	7.04	.439	33.69
25	1197	14.45	.472	59.84
50	1682	24.00	.424	84.09
75	1907	31.37	.372	95.34

EXPON ($\lambda = 1.75$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	498	3.53	.337	24.89
6	660	4.74	.381	32.99
8	823	5.84	.399	41.14
10	972	6.88	.410	48.59
25	1657	13.59	.414	82.84
50	1957	20.36	.307	97.84
75	1998	24.46	.230	99.89

EXPON ($\lambda = 2$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	617	3.51	.326	30.84
6	868	4.71	.370	43.39
8	1083	5.77	.387	54.14
10	1247	6.74	.386	62.34
25	1878	12.40	.337	93.89
50	1997	17.22	.214	99.84
75	2000	20.51	.160	100.00

B. DISCUSSION

The actual probability of crossing appeared to be consistently smaller than the theoretical probability under the null hypothesis ($\lambda = 1$). For N of fifty, and over, the agreement appeared to be reasonably good. Moreover, the probability of rejecting the null hypothesis (power) was quite high when $\lambda = 1.5$ and N was at least twenty-five. This would not be an unreasonable sample size for the transistor testing illustration, but would undoubtedly by large if the procedure were applied to check entire systems.

Notice that an empirically generated correction could be derived to adjust the size of the null hypothesis probability. Consider, for example, the $N = 4$, $\lambda = 1.25$ element of the nominal five per cent table: the crossing per cent (estimated true probability of crossing) is nearly five per cent. Under the null hypothesis, the simulation crossing probability was computed at 2.3 per cent. Now, if the null hypothesis prevailed, and if each failure time were multiplied by $(1.25)^{-1} = 0.8$, then five percent boundary crossings would be realized. On the other hand, if $\lambda > 1$, and each failure time was multiplied by 0.8, but still applied against the theoretical derived boundary for $\lambda = 1$, the power of the procedure would be further enhanced. This empirical adjustment has been left for more extensive development, e.g. -- for different sample sizes-- in the future. It seems clear that an alternate approach involves an adjustment of the boundary, itself.

C. EXPERIMENT 2.

Using the same procedure as in Experiment 1, we explore the effect of $\alpha \neq \beta$, but with $\alpha\beta$ = constant. We tried the following four combinations: (1) $\alpha = 1.5\beta$, (2) $\beta = 1.5\alpha$, (3) $\alpha = 2\beta$, (4) $\beta = 2\alpha$. The results are summarized in the next tables.

TABLE III

5% Probability of a Crossing
$\alpha = 1.49893 \quad \beta = .99929$

EXPON (λ =.5)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	2	4.00	.156	.10
6	1	6.00	.476	.05
8	3	7.33	.564	.15
10	0	0.00	.000	.00
25	0	0.00	.000	.00
50	0	0.00	.000	.00
75	0	0.00	.000	.00

EXPON (λ = 1)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	17	4.00	.297	.85
6	36	5.61	.461	1.80
8	44	6.95	.492	2.20
10	49	8.20	.504	2.45
25	49	17.28	.589	2.45
50	70	32.91	.697	3.50
75	91	47.46	.706	4.55

EXPON (λ = 1.25)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	36	4.00	.309	1.80
6	97	5.59	.457	4.85
8	118	7.00	.493	5.90
10	147	8.34	.529	7.35
25	340	18.20	.668	16.99
50	636	32.99	.697	31.79
75	871	46.54	.692	43.54

TABLE III (continued)

EXPON (λ = 1.5)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	76	4.00	.316	3.80
6	161	5.59	.428	8.04
8	232	7.03	.495	11.60
10	324	8.36	.531	16.19
25	857	17.47	.603	42.84
50	1472	29.77	.562	73.59
75	1799	40.34	.521	89.94

EXPON (λ = 1.75)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	126	4.00	.314	6.30
6	254	5.55	.415	12.70
8	401	6.91	.470	20.04
10	560	8.27	.511	27.99
25	1381	16.67	.538	69.04
50	1897	26.42	.436	94.84
75	1990	32.57	.334	99.49

EXPON (λ = 2)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	189	4.00	.310	9.44
6	402	5.50	.409	20.09
8	594	6.85	.454	29.70
10	787	8.14	.481	39.34
25	1733	15.70	.462	86.64
50	1987	22.91	.312	99.34
75	2000	27.50	.229	100.00

TABLE IV

5% Probability of a Crossing
$\alpha = .99929 \quad \beta = 1.49893$

EXPON ($\lambda = .5$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	10	3.10	.203	0.50
6	4	4.50	.273	0.20
8	10	5.70	.331	0.50
10	3	5.66	.240	0.15
25	0	0.00	.000	0.00
50	0	0.00	.000	0.00
75	0	0.00	.000	0.00

EXPON ($\lambda = 1$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	80	3.30	.241	4.00
6	99	4.48	.301	4.95
8	90	5.44	.291	4.50
10	101	6.27	.304	5.05
25	91	12.68	.331	4.55
50	99	23.08	.374	4.95
75	117	34.13	.411	5.85

EXPON ($\lambda = 1.25$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	162	3.38	.263	8.10
6	198	4.52	.301	9.90
8	214	5.61	.323	10.70
10	242	6.57	.334	12.10
25	392	13.78	.392	19.59
50	628	24.36	.413	31.39
75	797	35.16	.434	39.84

TABLE IV (continued)

EXPON ($\lambda = 1.5$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	248	3.42	.268	12.39
6	334	4.47	.291	16.70
8	400	5.59	.319	19.99
10	455	6.58	.333	22.74
25	889	13.77	.396	44.45
50	1406	23.44	.386	70.29
75	1720	31.24	.357	85.99

EXPON ($\lambda = 1.75$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	346	3.43	.264	17.29
6	477	4.47	.287	23.84
8	588	5.55	.313	29.39
10	729	6.60	.335	36.45
25	1368	13.16	.361	68.39
50	1867	20.58	.303	93.34
75	1979	24.88	.235	98.95

EXPON ($\lambda = 2$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	466	3.45	.268	23.29
6	644	4.51	.293	32.19
8	806	5.53	.307	40.29
10	977	6.53	.323	48.84
25	1714	12.37	.316	85.70
50	1980	17.39	.216	98.99
75	not run			

TABLE V

5% Probability of a Crossing
$\alpha = 1.73082 \quad \beta = .86541$

EXPON (λ = .5)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	2	4.00	.156	0.10
6	1	6.00	.476	0.05
8	1	8.00	.649	0.05
10	0	0.00	.000	0.00
25	0	0.00	.000	0.00
50	0	0.00	.000	0.00
75	0	0.00	.000	0.00

EXPON (λ = 1)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	4	4.00	.169	0.20
6	17	5.88	.445	0.85
8	28	7.50	.564	1.40
10	34	8.76	.567	1.70
25	39	18.92	.703	1.95
50	59	36.86	.858	2.95
75	85	52.07	.845	4.25

EXPON (λ = 1.25)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	11	4.00	.207	0.55
6	48	5.87	.460	2.40
8	70	7.40	.538	3.50
10	103	8.84	.583	5.15
25	308	19.84	.805	15.40
50	612	35.67	.808	30.59
75	851	50.50	.803	42.54

TABLE V (continued)

EXPON ($\lambda = 1.5$)

N	NC	$\overline{A}(t)$	$\overline{C}(t)$	f_N
4	17	4.00	.198	0.85
6	94	5.86	.441	4.70
8	161	7.40	.534	8.04
10	253	8.89	.595	12.64
25	804	18.82	.691	40.19
50	1463	32.42	.651	73.14
75	1802	43.71	.589	90.09

EXPON ($\lambda = 1.75$)

N	NC	$\overline{A}(t)$	$\overline{C}(t)$	f_N
4	28	4.00	.204	1.40
6	169	5.81	.427	8.45
8	286	7.33	.507	14.29
10	447	8.83	.571	22.34
25	1327	18.03	.620	66.34
50	1895	28.73	.492	94.74
75	1987	35.92	.381	99.34

EXPON ($\lambda = 2$)

N	NC	$\overline{A}(t)$	$\overline{C}(t)$	f_N
4	54	4.00	.216	2.70
6	239	5.81	.418	11.95
8	447	7.29	.497	22.34
10	666	8.70	.538	33.29
25	1724	17.12	.535	86.20
50	1985	25.28	.358	99.24
75	2000	30.74	.265	100.00

TABLE VI

5% Probability of a Crossing
$\alpha = .86541 \quad \beta = 1.73082$

EXPON ($\lambda = .5$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	14	2.78	.151	0.70
6	6	4.16	.258	0.30
8	11	5.54	.314	0.55
10	5	4.80	.176	0.25
25	0	0.00	.000	0.00
50	1	13.00	.130	0.05
75	1	18.00	.130	0.05

EXPON ($\lambda = 1$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	97	3.02	.203	4.85
6	117	3.95	.230	5.85
8	106	5.00	.254	5.30
10	115	5.60	.239	5.75
25	105	11.48	.281	5.25
50	119	20.31	.304	5.95
75	122	30.05	.336	6.10

EXPON ($\lambda = 1.25$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	192	3.10	.225	9.60
6	228	4.13	.253	11.40
8	245	5.10	.274	12.24
10	265	5.98	.279	13.25
25	400	12.42	.328	19.99
50	600	21.42	.337	29.99
75	777	31.81	.373	38.84

TABLE VI (continued)

EXPON ($\lambda = 1.5$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	299	3.15	.232	14.95
6	361	4.14	.253	18.04
8	422	5.08	.269	21.09
10	481	6.08	.287	24.04
25	859	12.28	.323	42.95
50	1363	21.33	.331	68.14
75	1679	28.53	.312	83.94

EXPON ($\lambda = 1.75$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	399	3.15	.226	19.94
6	504	4.17	.254	25.20
8	613	5.07	.263	30.64
10	728	6.03	.281	36.39
25	1320	11.96	.309	65.99
50	1843	19.00	.271	92.14
75	1972	22.88	.212	98.59

EXPON ($\lambda = 2$)

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	526	3.15	.229	26.29
6	682	4.15	.250	34.09
8	837	5.06	.262	41.84
10	991	6.01	.280	49.54
25	1703	11.36	.278	85.14
50	1976	15.98	.194	98.79
75	not run			

D. DISCUSSION

The tendency of the theory to understate the crossing probability, noted $\alpha = \beta$, is even more pronounced when $\alpha = 1.5\beta$. However, when $\beta = 1.5\alpha$ (TABLE IV) the understatement error nearly vanishes, and the test is very close to being exact (i.e.- crossings occur with probability about five percent) for all values of N checked. According to TABLE VI, the probability of rejection is slightly overstated when $\beta = 2\alpha$, for all but N = 4. To within limits of sampling accuracy, it appears that β equalling 1.65α will give a very nearly correct answer at $\delta = 0.05$. Further sampling will pin this down more precisely. A theoretical explanation would be most interesting, for the asymptotic (large N) mathematics detects no difference in situations for which $\alpha\beta$ = constant.

The next set of experiments was conducted in order to investigate our procedure's ability to detect departures from the null (exponential, $\lambda = 1$) hypothesis.

E. EXPERIMENT 3.

Exploration of the crossing probability when the boundary reflected the null hypothesis (exponential with $\lambda = \lambda_0 = 1$, $\alpha = \beta$, nominal crossing probability five percent), but samples were drawn from a special long-tailed distribution with a Pareto-like appearance:

$$F_p(x) = \frac{ax}{1+ax}, \; x \geq 0. \qquad (4.1)$$

In order to give a fair comparison, the parameter a was chosen so as to match the medians of F_p and the null distribution; thus $a = (\log_e 2)^{-1}$.

A rationale for the choice of (4.1) is that such a distribution appears very similar to the exponential, but has a decreasing failure rate and a bias toward early failures. The results are recorded in TABLE VII.

TABLE VII

5% Probability of a Crossing
$\alpha = \beta = 1.22388$

PARETO

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	79	3.50	.253	3.95
6	97	4.82	.319	4.85
8	91	5.61	.274	4.55
10	104	6.70	.309	5.20
25	133	12.30	.277	6.64
50	209	20.00	.257	10.45
75	218	26.65	.242	10.90

F. DISCUSSION.

Our sampling experiment indicates that the sequential procedure under study has very little power against the "Pareto tail" alternative (4.1) under the particular choice of parameters, at least for small N. Actually, the probability of rejection is somewhat larger than for the corresponding exponential. It seems likely that the relative prevalence of small observations accounts for the increased crossing probability, and also for the relatively short expected time to cross, given that a crossing occurred.

G. EXPERIMENT 4.

The procedure used in Experiment 3 was repeated, except that samples were drawn from a shorter-tailed (Weibull, with increasing hazard rate) distribution:

$$F_W(x) = 1-e^{-ax^2} \quad . \qquad (4.2)$$

Again, the parameter a was fitted by matching medians of the exponential

to (4.2); by coincidence, $a = (\log_e 2)^{-1}$ once again-- an accident of parameterization. The results are recorded in TABLE VIII.

TABLE VIII

5% Probability of a Crossing
$\alpha = \beta = 1.22388$

N	NC	$\bar{A}(t)$	$\bar{C}(t)$	f_N
4	13	4.00	.425	0.65
6	38	5.89	.610	1.90
8	58	7.72	.702	2.90
10	91	9.46	.778	4.55
25	805	22.78	1.078	40.24
50	1894	41.88	1.091	94.69
75	1996	58.30	1.009	99.79

H. DISCUSSION.

The sampling experiment suggests that the power of our sequential procedure is quite unsatisfactory against the particular Weibull alternative examined for small N, but the power rapidly increases with N. Scrutinizing of the density of (4.2) indicates that there are relatively few "early failures", while a concentration of "wearouts" probably accounts for the rather long conditional expectation of the time to hit the boundary (reject the null hypothesis).

COMPUTER OUTPUT

```
N=   4     NR=2000     ALFA=   1.22388     BETA=   1.22388     STP=1000.00000

EXPONENTIAL  LAMBDA=   1.00000

NUMBER OF HITS=  46.00000     PERCENT=   2.30000

NUMBER OF NONHITS=    7937.0

AVERAGE AN(T) IS   3.63043

AVERAGE HIT TIME IS    0.28702

PRESET TERMINATION TIME=1000.00000
```

```
N=    6     NR=2000     ALFA=    1.22388     BETA=    1.22388     STP=1000.00000

EXPONENTIAL  LAMBDA=    1.00000

NUMBER OF HITS=  69.00000     PERCENT=    3.45000

NUMBER OF NONHITS=    11867.0

AVERAGE AN(T) IS    5.07246

AVERAGE HIT TIME IS    0.39110

PRESET TERMINATION TIME=1000.00000
```

```
N=    8      NR=2000      ALFA=    1.22388      BETA=    1.22388      STP=1000.00000

EXPONENTIAL  LAMBDA=    1.00000

NUMBER OF HITS=  66.00000      PERCENT=    3.30000

NUMBER OF NONHITS=   15822.0

AVERAGE AN(T) IS    6.30303

AVERAGE HIT TIME IS     0.40292

PRESET TERMINATION TIME=1000.00000
```

```
N=  10     NR=2000     ALFA=   1.22388     BETA=   1.22388     STP=1000.00000

EXPONENTIAL  LAMBDA=   1.00000

NUMBER OF HITS=  70.00000     PERCENT=   3.50000

NUMBER OF NONHITS=   19746.0

AVERAGE AN(T) IS   7.37143

AVERAGE HIT TIME IS    0.41607

PRESET TERMINATION TIME=1000.00000
```

```
N=  25     NR=2000     ALFA=   1.22388     BETA=   1.22388     STP=1000.00000

EXPONENTIAL  LAMBDA=   1.00000

NUMBER OF HITS=  73.00000     PERCENT=   3.65000

NUMBER OF NONHITS=   49211.0

AVERAGE AN(T) IS  15.19178

AVERAGE HIT TIME IS    0.46166

PRESET TERMINATION TIME=1000.00000
```

```
N=  50     NR=2000     ALFA=   1.22388     BETA=   1.22388     STP=1000.00000

EXPONENTIAL  LAMBDA=   1.00000

NUMBER OF HITS=  85.00000     PERCENT=   4.25000

NUMBER OF NONHITS=   98006.0

AVERAGE AN(T) IS  27.54117

AVERAGE HIT TIME IS    0.50430

PRESET TERMINATION TIME=1000.00000
```

```
N=  75     NR= 2000     ALFA=  1.22388     BETA=  1.22388     STP= 1000.00000

EXPONENTIAL

NUMBER OF HITS= 104.00000     PERCENT=  5.20000

NUMBER OF NONHITS=  146509.0

AVERAGE AN(T) IS  42.43268

AVERAGE HIT TIME IS    0.58683

PRESET TERMINATION TIME= 1000.00000
```

COMPUTER PROGRAM

```
C
C
C
C                 EXPONENTIAL BOUNDRY AND EXPONENTIAL ARRIVALS
C
C
//GWSI14 JOB (1155,0906FT,RL14),'GWINN',TIME=30
// EXEC FORTCLG,REGION.GO=100K
//FORT.SYSIN DD *
      DIMENSION TO(100),R(100)                                          LAM00010
      DO 650 II=1,6                                                     LAM00015
      READ(5,4) XLM                                                     LAM00020
    4 FORMAT(F10.5)                                                     LAM00030
      ZLM=1./XLM                                                        LAM00040
    2 ZERO=0                                                            LAM00050
      ONE=0                                                             LAM00060
C                                                                       LAM00070
C                                                                       LAM00080
C   GENERATE(N) UNIFORM RANDOM NUMBERS                                  LAM00090
C                                                                       LAM00100
      READ(5,3)N,NR,ALFA,BETA,STP                                       LAM00110
    3 FORMAT(2I4,3F10.5)                                                LAM00120
      IF(N.EQ.9999) GO TO 650                                           LAM00130
      WRITE(6,702)N,NR,ALFA,BETA,STP,XLM                                LAM00140
      NS=N                                                              LAM00150
      CALL OVFLOW                                                       LAM00160
      IX=547626765                                                      LAM00170
C                                                                       LAM00180
C                                                                       LAM00190
C   GENERATE EXPONENTIAL ARRIVAL TIMES T(J)                             LAM00200
C                                                                       LAM00210
      D=0.                                                              LAM00220
      RNS=0.                                                            LAM00230
      H=0.                                                              LAM00240
      G=0.                                                              LAM00250
      K=0                                                               LAM00260
    5 DO 500 M=1,NR                                                     LAM00270
      L=0                                                               LAM00280
      CALL RANDOM(IX,R,NS)                                              LAM00290
      DO 10 J=1,N                                                       LAM00300
      L=L+1                                                             LAM00310
      TO(J)=-ZLM*ALOG(R(L))                                             LAM00320
   10 CONTINUE                                                          LAM00330
C                                                                       LAM00340
C                                                                       LAM00350
C   SORT ROUTINE                                                        LAM00360
C                                                                       LAM00370
      LIM=N-1                                                           LAM00380
```

```
  101 INT=1
      DO 200 I=1,LIM
      IF(TO(I+1).GE.TO(I)) GO TO 200
      TEMP=TO(I+1)
      TO(I+1)=TO(I)
      TO(I)=TEMP
      INT=I
  200 CONTINUE
      IF(INT.EQ.1) GO TO 300
      LIM=INT-1
      GO TO 101
  300 CONTINUE
C
C
C  GENERATE AND COMPARE AN(TO) AND B(TO)
C
      RN=N
      SQR=SQRT(RN)
      DO 402 I=1,N
      IF(TO(I).GE.STP) GO TO 403
      AN=I
      F=1.-EXP(-TO(I))
      B=RN*F+SQR*(ALFA*(1.-F)+BETA*F)
      IF(AN.GT.B) GO TO 401
      ZERO=ZERO+1
      GO TO 402
  401 ONE=ONE+1
      G=G+1.
      D=D+TO(I)
      H=H+AN
      GO TO 500
  402 CONTINUE
      GO TO 500
  403 CONTINUE
C
C
C  OUTPUT SECTION
C
  500 CONTINUE
      RNS=NR
      HIT=(ONE/RNS)*100.
      IF(G.EQ.0.) GO TO 600
      D=D/G
      H=H/G
  600 WRITE(6,700)ONE,HIT
      WRITE(6,703)ZERO
      WRITE(6,706)H
      WRITE(6,708)D
```

```
      WRITE(6,707)STP                                                   LAM00870
      GO TO 2                                                           LAM00880
  650 CONTINUE                                                          LAM00885
  700 FORMAT(/// 20X,'NUMBER OF HITS=',F10.5,5X,'PERCENT=',F10.5,/)     LAM00890
  702 FORMAT('1'/////////,20X,'N=',I4,5X,'NR=',I4,5X,'ALFA=',F10.5,5X,'BLAM00900
     1ETA=',F10.5,5X,'STP=',F10.5,////20X,'EXPONENTIAL  LAMBDA=',F10.5) LAM00910
  703 FORMAT(//20X,'NUMBER OF NONHITS=',F10.1)                          LAM00920
  706 FORMAT(/// 20X,'AVERAGE AN(T) IS',F10.5)                          LAM00930
  707 FORMAT(///20X,'PRESET TERMINATION TIME=',F10.5///)                LAM00940
  708 FORMAT(///20X,'AVERAGE HIT TIME IS ',F10.5)                       LAM00950
  999 STOP                                                              LAM00960
      END                                                               LAM00970
C
C               DATA FOLLOWS
C
//GO.SYSIN DD *
.5
   42000     .89706    .897061000.
   62000     .89706    .897061000.
   82000     .89706    .897061000.
  102000     .89706    .897061000.
  252000     .89706    .897061000.
  502000     .89706    .897061000.
  752000     .89706    .897061000.
9990000000000000000000000000000000000
1.
   42000     .89706    .897061000.
   62000     .89706    .897061000.
   82000     .89706    .897061000.
  102000     .89706    .897061000.
  252000     .89706    .897061000.
  502000     .89706    .897061000.
  752000     .89706    .897061000.
9990000000000000000000000000000000000
1.25
   42000     .89706    .897061000.
   62000     .89706    .897061000.
   82000     .89706    .897061000.
  102000     .89706    .897061000.
  252000     .89706    .897061000.
  502000     .89706    .897061000.
  752000     .89706    .897061000.
9990000000000000000000000000000000000
1.5
   42000     .89706    .897061000.
   62000     .89706    .897061000.
   82000     .89706    .897061000.
  102000     .89706    .897061000.
```

```
  252000     .89706     .897061000.
  502000     .89706     .897061000.
  752000     .89706     .897061000.
9999000000000000000000000000000000000000
1.75
   42000     .89706     .897061000.
   62000     .89706     .897061000.
   82000     .89706     .897061000.
  102000     .89706     .897061000.
  252000     .89706     .897061000.
  502000     .89706     .897061000.
  752000     .89706     .897061000.
9999000000000000000000000000000000000000
2.
   42000     .89706     .897061000.
   62000     .89706     .897061000.
   82000     .89706     .897061000.
  102000     .89706     .897061000.
  252000     .89706     .897061000.
  502000     .89706     .897061000.
  752000     .89706     .897061000.
9999000000000000000000000000000000000000
```

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13. ABSTRACT

An approximate boundary for a finite sample sequential decision process is presented (without derivation). By means of Computer Simulation, properties of the process are checked; the power of the test is thus determined for selected alternative hypotheses.

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